

# Explicit solution for vibrating bar with viscous boundaries and internal damper

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## Abstract

We investigate longitudinal vibrations of a bar subjected to viscous boundary conditions at each end, and an internal damper at an arbitrary point along the bar's length. The system is described by four independent parameters and exhibits a variety of behaviors including rigid motion, super stability/instability and zero damping. The solution is obtained by applying the Laplace transform to the equation of motion and computing the Green's function of the transformed problem. This leads to an unconventional eigenvalue-like problem with the spectral variable in the boundary conditions. The eigenmodes of the problem are necessarily complex-valued and are not orthogonal in the usual inner product. Nonetheless, in generic cases we obtain an explicit eigenmode expansion for the response of the bar to initial conditions and external force. For some special values of parameters the system of eigenmodes may become incomplete, or no non-trivial eigenmodes may exist at all. We thoroughly analyze physical and mathematical reasons for this behavior and explicitly identify the corresponding parameter values. In particular, when no eigenmodes exist, we obtain closed form solutions. Theoretical analysis is complemented by numerical simulations, and analytic solutions are compared to computations using finite elements.

**Keywords:** longitudinal vibrations, viscous boundary conditions, modal decomposition, vibratory response.

# 1 Introduction

In this paper we analyze longitudinal vibrations of a bar with dampers attached at each end as well as at an internal point of the bar. This type of problem occurs in modeling structures containing shock absorbers and in control of continuous structures with discrete elements. Mathematically, the problem reduces to solving the wave equation modified by a Dirac delta term with viscous boundary conditions. When the boundary conditions are classical, e.g. the ends are free or clamped, separation of variables or Laplace transforms reduce the situation to a boundary eigenvalue problem for a second-order ODE called the Sturm-Liouville problem. These problems are self-adjoint and admit a complete system of orthogonal eigenmodes the solution can be expanded into with coefficients determined from initial values using the orthogonality.

When viscous boundaries are present the Laplace transform leads to a boundary value problem with the spectral parameter entering boundary conditions. One still gets a system of eigenmodes, but they are not orthogonal in the usual inner product, and the eigenvalues are general complex numbers reflecting the non-self-adjoint nature of the problem. For some critical values of damping parameters the system of eigenmodes may not be complete, and even when it is finding the expansion coefficients in terms of initial data is non-trivial because the eigenmodes are not orthogonal. Although studied by mathematicians [1, 2] such problems and their properties are rather sparsely treated in the engineering literature, nevertheless see [3, 4, 5, 6] and example 4 in [7, chap 4]. Hull [3] was first to treat a bar with a viscous boundary at one end and clamped at the other, but he utilized a non-standard approach to decoupling the equations of motions and provided a response only for a harmonic driving force. Udawadia [6] appears to be the first to provide a complete closed form solution to this problem via Laplace transform.

Adding an internal damper at an arbitrary point of the bar, as we do in this paper, significantly complicates a closed form solution to the problem. In particular, it is no longer possible to find analytic formulas for the eigenvalues since their determination depends on solving algebraic equations of arbitrarily high degree. However, if the eigenvalues are found numerically the solution can be written explicitly in a closed form. We obtain the analytical solution by taking the Laplace transform and finding the Green's function for the resulting boundary eigenvalue problem. In our analysis we are able to take advantage of general mathematical results that simplify calculations considerably. For example, although we do find a cumbersome explicit formula for the Green's function it is not necessary to find the expansion of its inverse Laplace transform, which depends on eigenmodes and eigenvalues only.

Moreover, many qualitative traits of the solution can be gleaned from the characteristic equation for eigenvalues directly without computing the vibratory response. Internal damper in a problem with free ends was considered in [8].

Behavior of the bar is controlled by four dimensionless parameters, the damping coefficients  $h_1$  and  $h_2$  at the left and right ends, the internal damping coefficient  $h_3$ , and the ratio  $a/L$  characterizing the position of the internal damper ( $a$  is the distance to the damper from the left end of the bar, and  $L$  is its length). Since the dimension of the parameter space is four it can not be easily visualized. As the parameters are varied, the bar exhibits a variety of behaviors including rigid motions, zero damping, super stability and instability. Although a four-dimensional diagram can not be drawn we give analytic conditions for all these types of behavior. Much of the unusual behavior is due to the fact that we do not restrict  $h_i$ 's to positive values they would take if the dampers are realized as dashpots. For the negative values we are dealing with so-called active dashpots, or rather 'pushpots', that add energy to the bar instead of damping it. Such discrete elements are sometimes used in control problems for continuous structures [6].

Perhaps the most striking observation is the extreme sensitivity of the eigenvalue distribution to the nature of the number  $a/L$ . When this number is rational the eigenvalues are generically distributed along  $p$  vertical lines in the complex plane, where  $p$  is the denominator of  $a/L$  in lowest terms. This significantly complicates expansion into eigenmodes since increasingly larger numbers of them have to be kept. When  $a/L$  is irrational this distribution appears random. Vibratory response on the other hand, is qualitatively insensitive to the placement of the internal damper, and in practice one may want to use ratios with small denominators like  $1/2, 1/3, 2/3$ , etc. A flip side of these observations is that while FEM produces good approximation for the vibratory response at least for short times it performs poorly in approximating the eigenvalues. In fact, it produces spurious eigenvalues with large real parts that do not converge to actual eigenvalues as the number of elements is increased. When the real parts of the spurious eigenvalues are positive FEM will lead to large errors in the vibratory response at large times.

We organize our presentation as follows. The next section gives the precise problem statement and describes our approach towards solving it and the main results obtained. In sections 3,4 we respectively derive analytic formulas for the eigenmodes, and reduce computing the eigenvalues to solving an algebraic equation for  $a/L$  rational. Section 5 discusses in more detail the case, when the damper is placed exactly in the middle of the bar, i.e.  $a/L = 1/2$ . We compute the eigenvalues explicitly and also give explicit conditions for the undamped behavior of the bar. The

Laplace transform of the Green's function of our problem is computed in Section 6 and we discuss its expansion into partial fractions. Under generic conditions such expansion exists and is easily Laplace inverted providing a convenient way for solving our initial-boundary problem. We also give formulas for special combinations of parameters when the Green's function can be inverted analytically. Section 7 presents theoretical analysis of eigenmode completeness for our problem, and discusses the physical meaning of critical cases when this completeness is lost. In section 8 we use the eigenmode expansion of the Green's function to write the vibratory response of the bar to initial data and external force. Section 8 compares analytic and FEM solutions in several parameter regimes and discusses spurious eigenvalues produced by FEM. Finally, we draw our conclusions in section 9. Appendix gives derivations of some formulas used in the main text.

## 2 Main results

We begin with the problem statement. Figure 1 depicts a bar of length  $L$  free to move horizontally suspended by two dampers at each end and by one at the distance  $a$  from its left end. Symbols  $\rho$ ,  $A$  and  $E$  represent the density of the bar, the constant cross-sectional area and its modulus of elasticity respectively, the wave speed along the bar is denoted  $c := (E/\rho)^{1/2}$ . Let  $c_1$ ,  $c_2$  and  $c_3$  be the damping coefficients of the left, right and internal dampers respectively, we set  $h_1 := \frac{c}{EA}c_1$ ,  $h_2 := \frac{c}{EA}c_2$  and  $h_3 := \frac{c}{2EA}c_3$  (the extra  $1/2$  simplifies some formulas). These  $h_i$  along with  $a/L$  are the dimensionless parameters that determine qualitative behavior of the bar. Since in our case the bar can move rigidly just as in the problem with free ends, we write the equation of motion in the absolute frame that remains at rest at all times. At  $t = 0$  the left end of the bar is assumed to be at the origin, and  $u(x, t)$  denotes the displacement of the point with initial coordinate  $x$  at time  $t$ , see Fig.1.

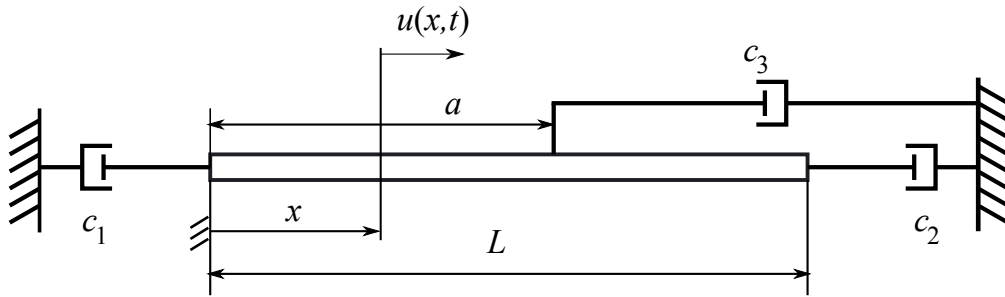


Figure 1: A bar with viscous ends and internal damper.

The system is described by a modified wave equation

$$u_{tt}(x, t) + 2h_3c\delta(x - a)u_t(x, t) - c^2u_{xx}(x, t) = p(x, t), \quad (1)$$

with the boundary conditions

$$u_x(0, t) - \frac{h_1}{c}u_t(0, t) = 0 \quad \text{and} \quad u_x(L, t) + \frac{h_2}{c}u_t(L, t) = 0. \quad (2)$$

Here  $p(x, t)$  is the external force per unit mass, and the subscripts  $x, t$  denote partial derivatives with respect to space and time. Given  $u(x, t)$  the solution in the frame that moves along with the left end of the bar is  $u(x, t) - u(0, t)$ .

To solve the problem we use Laplace transforms. Setting the external force and initial data to zero and taking Laplace transform we get a homogeneous boundary problem for  $U(x, s) := \mathcal{L}[u(x, t)]$

$$\frac{s^2}{c^2}U(x, s) + 2h_3\frac{s}{c}\delta(x - a)U(x, s) - U_{xx}(x, s) = 0, \quad (3)$$

$$U_x(0, s) - h_1\frac{s}{c}U(0, s) = 0 \quad \text{and} \quad U_x(L, s) + h_2\frac{s}{c}U(L, s) = 0. \quad (4)$$

We will explicitly compute the Green's function  $G(x, \xi, s)$  for this problem and use it to solve the original initial-boundary problem.

Unfortunately, the inverse Laplace transform of  $G(x, \xi, s)$  can not be computed in closed form except in special cases. To invert the Laplace transform for  $G(x, \xi, s)$  we use the spectral method. Note that one gets the same boundary problem by separating variables and looking for solutions of the form  $u(x, t) = \varphi_a(x, s)e^{st}$ , where  $s$  is the spectral parameter. System (3),(4) is almost an ordinary boundary eigenvalue problem except for the presence of  $s$  in the boundary conditions. One expects that it is solvable only for special values  $s_n$ , the eigenvalues, with  $\varphi_a(x, s_n)$  being the vibrational eigenmodes of the bar. This is indeed the case, and for  $a/L = q/p$  (with integers  $q, p$  in lowest terms) they can be grouped into  $p$  series  $s_n^{(k)}$ , one for each root of a degree  $p$  algebraic equation. These roots are the only quantities to be computed numerically. Once they are determined,  $s_n^{(k)}$  and the eigenmodes can be written explicitly.

We then show that under generic conditions the Green's function  $\mathcal{L}^{-1}[G(x, \xi, s)]$  can be expanded into a series over the eigenmodes and give an explicit formula for the expansion coefficients  $A_n^{(k)}$ . With all the pieces in place we derive our main result, a series solution for the vibratory response of the bar with initial data  $u(x, 0)$ ,  $\dot{u}(x, 0)$  subjected to the external force  $p(x, t)$ :

$$\begin{aligned}
u(x, t) = & \frac{1}{h_1 + h_2 + 2h_3} \left[ h_1 u(0, 0) + h_2 u(L, 0) + 2h_3 u(a, 0) + \frac{1}{c} \int_0^L \left[ \dot{u}(\xi, 0) + \int_0^t p(\xi, \tau) d\tau \right] d\xi \right] \\
& + \sum_{k=1}^p \sum_{n=-\infty}^{\infty} \frac{\varphi_a(x, s_n^{(k)}) e^{s_n^{(k)} t}}{c^2 A_n^{(k)}} \left[ \left[ c h_1 u(0, 0) + c h_2 u(L, 0) \varphi_a(L, s_n^{(k)}) + 2c h_3 u(a, 0) \varphi_a(a, s_n^{(k)}) \right] \right. \\
& \left. + \int_0^L \left[ s_n^{(k)} u(\xi, 0) + \dot{u}(\xi, 0) + \int_0^t p(\xi, \tau) e^{-s_n^{(k)} \tau} d\tau \right] \varphi_a(\xi, s_n^{(k)}) d\xi \right]. \quad (5)
\end{aligned}$$

The other major result is a complete analysis of critical behavior. Namely, we prove that the eigenmodes and associated modes are sufficient to expand the Green's function if and only if  $h_i \neq \pm 1$ . When  $h_2 = 1, h_3 = 0$  or  $h_1 = h_2 = 1$  we derive alternative solutions in closed form.

### 3 Eigenmodes

In this section we compute the eigenmodes of our problem analytically. To aid notation and understanding we first compute the eigenmodes for the simpler system without the internal damper ( $h_3 = 0$ ). Define  $\varphi(x, s)$ ,  $\psi(x, s)$  as solutions to  $\frac{s^2}{c^2} U(x, s) - U_{xx}(x, s) = 0$  satisfying only the left and the right boundary condition from Eq.(4) respectively. This defines them up to a constant multiple and we make them unique by normalizing  $\varphi(0, s) = 1 = \psi(L, s)$ . One easily finds

$$\varphi(x, s) = \cosh\left(\frac{sx}{c}\right) + h_1 \sinh\left(\frac{sx}{c}\right) = \frac{1}{2} \left[ (1 + h_1) e^{\frac{sx}{c}} + (1 - h_1) e^{-\frac{sx}{c}} \right] \quad (6)$$

$$\psi(x, s) = \cosh\left(\frac{s(L-x)}{c}\right) + h_2 \sinh\left(\frac{s(L-x)}{c}\right) = \frac{1}{2} \left[ (1 + h_2) e^{\frac{s(L-x)}{c}} + (1 - h_2) e^{-\frac{s(L-x)}{c}} \right]. \quad (7)$$

By construction, any solution to  $\frac{s^2}{c^2} U - U_{xx} = 0$  satisfying the left (right) boundary condition must be a multiple of  $\varphi(x, s)$  ( $\psi(x, s)$ ). Since the eigenmodes are supposed to satisfy both we conclude that they only exist for values

of  $s$  that make  $\varphi(x, s)$  and  $\psi(x, s)$  proportional. For such values either one of them can be taken as the eigenmode.

Proportionality happens if and only if the Wronskian  $W[\varphi, \psi] := \begin{vmatrix} \varphi & \psi \\ \varphi'_x & \psi'_x \end{vmatrix}$  is zero giving us an equation for eigenvalues  $W[\varphi, \psi] = 0$ . For future reference we denote

$$\begin{aligned} \Delta(s) &:= -\frac{c}{s} W[\varphi, \psi] = (1 + h_1 h_2) \sinh\left(\frac{sL}{c}\right) + (h_1 + h_2) \cosh\left(\frac{sL}{c}\right) \\ &= \frac{1}{2} \left[ (1 + h_1)(1 + h_2) e^{\frac{sL}{c}} - (1 - h_1)(1 - h_2) e^{-\frac{sL}{c}} \right], \end{aligned} \quad (8)$$

and remark that the eigenvalues other than  $s = 0$  satisfy  $\Delta(s) = 0$  or

$$e^{\frac{2sL}{c}} = \frac{(1 - h_1)(1 - h_2)}{(1 + h_1)(1 + h_2)}.$$

The latter can be solved explicitly:

$$s_n = \frac{c}{2L} \left[ \ln \left| \frac{(1 - h_1)(1 - h_2)}{(1 + h_1)(1 + h_2)} \right| + i \left( \text{Arg} \left( \frac{(1 - h_1)(1 - h_2)}{(1 + h_1)(1 + h_2)} \right) + 2\pi n \right) \right], \quad n \in \mathbb{Z}. \quad (9)$$

Substituting  $s_n$  into Eq.(6) gives the eigenmodes  $\varphi(x, s_n)$ , also explicitly.

We now apply the same scheme to Eq.(3) when  $h_3 \neq 0$ . As above, denote by  $\varphi_a(x, s)$  ( $\psi_a(x, s)$ ) solutions to Eq.(3) satisfying the left (right) boundary condition and normalized to be 1 at the corresponding boundary. Consider  $\varphi_a$  first. Since the damper at  $x = a$  does not affect the equation on  $[0, a)$  we have  $\varphi_a(x, s) = \varphi(x, s)$  on this interval.

For  $x > a$  our  $\varphi_a$  again satisfies  $\frac{s^2}{c^2} U - U_{xx} = 0$  but there must be a jump in its first derivative at  $a$  to produce the  $2h_3 \frac{s}{c} \delta(x - a) U$  term in Eq.(3). Along with continuity at  $a$  we have  $\varphi_a(a, s) - \varphi(a, s) = 0$  and  $\varphi'_a(a, s) - \varphi'(a, s) = 2h_3 \frac{s}{c} \varphi(a, s)$ . Since the difference  $\varphi_a - \varphi$  also satisfies  $\frac{s^2}{c^2} U - U_{xx} = 0$  we see by inspection that  $\varphi_a(x, s) = \varphi(x, s) + 2h_3 \varphi(a, s) \sinh\left(\frac{s(x-a)}{c}\right)$  for  $x > a$ .

One can compute  $\psi_a$  analogously or notice that by symmetry it can be obtained from  $\varphi_a$  by changing  $x$  to  $L - x$ ,  $a$  to  $L - a$ , and  $h_1$  to  $h_2$ . With the help of the unit step Heaviside function  $H(x) := \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$  the  $x < a$  and

$x > a$  cases can be unified into

$$\varphi_a(x, s) = \varphi(x, s) + 2h_3 H(x - a) \varphi(a, s) \sinh\left(\frac{s(x - a)}{c}\right) \quad (10)$$

$$\psi_a(x, s) = \psi(x, s) + 2h_3 H(a - x) \psi(a, s) \sinh\left(\frac{s(a - x)}{c}\right) \quad (11)$$

To further aid in notation we define

$$\Delta_a(s) = -\frac{c}{s} W[\varphi_a, \psi_a] = \Delta(s) + 2h_3 \varphi(a, s) \psi(a, s) \quad (12)$$

and compute explicitly

$$\begin{aligned} \Delta_a(s) = & (1 + h_1 h_2 + h_3(h_1 + h_2)) \sinh\left(\frac{sL}{c}\right) + (h_1 + h_2 + h_3(1 + h_1 h_2)) \cosh\left(\frac{sL}{c}\right) \\ & + h_3(h_2 - h_1) \sinh\left(\frac{s(L - a)}{c}\right) + h_3(1 - h_1 h_2) \cosh\left(\frac{s(L - a)}{c}\right). \end{aligned} \quad (13)$$

We will mostly use the exponential form of this expression

$$\begin{aligned} \Delta_a(s) = & \frac{1}{2} \left[ (1 + h_1)(1 + h_2)(1 + h_3) e^{\frac{sL}{c}} - (1 - h_1)(1 - h_2)(1 - h_3) e^{-\frac{sL}{c}} \right. \\ & \left. + (1 - h_1)(1 + h_2) h_3 e^{\frac{s(L - 2a)}{c}} + (1 + h_1)(1 - h_2) h_3 e^{-\frac{s(L - 2a)}{c}} \right]. \end{aligned} \quad (14)$$

Eigenvalues are the solutions to  $s\Delta_a(s) = 0$ , but  $\Delta_a(s)$  contains four distinct exponents and  $\Delta_a(s) = 0$  in general can not be solved explicitly. Still, if the zeros are found numerically one can get explicit expressions for the eigenmodes from Eq.(10). Note that  $s = 0$  corresponds to a rigid displacement of the bar since  $\varphi_a(x, 0) = 1$ . We address computation of other eigenvalues in the next section.

## 4 Eigenvalues

We now need to compute the eigenvalues, i.e. solve the characteristic equation  $s\Delta_a(s) = 0$ . As we saw in the previous section, when there is no internal damper the answer is given explicitly by Eq.(9). This is possible because



only two different exponents appear in  $\Delta(s)$ . However, in general  $\Delta_a(s)$  contains four different exponents and there is no analytic formula for its zeros. Nonetheless, it is possible to reduce this transcendental equation to solving an algebraic one. Let us multiply the exponential form of  $\Delta_a$  by  $e^{\frac{sL}{c}}$  and set:

$$A_1 = (1+h_1)(1+h_2)(1+h_3), \quad A_2 = h_3(1-h_1)(1+h_2), \quad A_3 = h_3(1+h_1)(1-h_2), \quad A_4 = -(1-h_1)(1-h_2)(1-h_3).$$

In terms of a new variable  $\xi := 2sL/c$  the equation becomes

$$A_1 e^\xi + A_2 e^{(1-\frac{a}{L})\xi} + A_3 e^{\frac{a}{L}\xi} + A_4 = 0. \quad (15)$$

The left hand side is a so-called exponential sum with real exponents, its zeros can be distributed in complicated ways. However, these complications only arise when the exponents in Eq.(15) are incommensurable, i.e.  $a/L$  is irrational [9, chap VI]. But irrational numbers can be approximated by rational numbers with arbitrary precision, so one can assume, for all practical purposes, that  $a/L$  is rational. We do so from now on.

Let  $a/L = q/p$  in lowest terms with positive integers  $p, q$ . A further substitution  $z = e^{\frac{\xi}{p}} = e^{\frac{2sL}{pc}}$  reduces Eq.(15) to an algebraic equation

$$A_1 z^p + A_2 z^{p-q} + A_3 z^q + A_4 = 0. \quad (16)$$

If  $A_1 \neq 0$  it has  $p$  roots  $z_1, z_2, \dots, z_p$  counting multiplicity, and we need not deal with infinitely many zeros of Eq.(15) directly. If moreover  $A_4 \neq 0$  none of these roots is 0. When  $A_1$  or  $A_4$  do vanish we get critical cases that require special treatment.

In general, roots  $z_k$  can not be found analytically as functions of  $h_i$ . However, there is a standard way of setting up a matrix with the characteristic equation Eq.(16) so that they can be found numerically as its eigenvalues using MATLAB or Maple. Once  $z_k$  are found, the eigenvalues can be expressed as (cf. Eq.(9))

$$s_n^{(k)} = \frac{pc}{2L} [\ln |z_k| + i(\text{Arg}(z_k) + 2\pi n)], \quad n \in \mathbb{Z}, \quad k = 1, \dots, p. \quad (17)$$

Fig.2 shows a typical distribution of eigenvalues, MATLAB function eig() was used to compute the roots  $z_k$ .

If all  $z_k$  are distinct then  $s_n^{(k)}$  fall into  $p$  infinite sequences with  $\text{Re}(s) = \frac{pc}{2L} \ln |z_k|$  equispaced along vertical lines

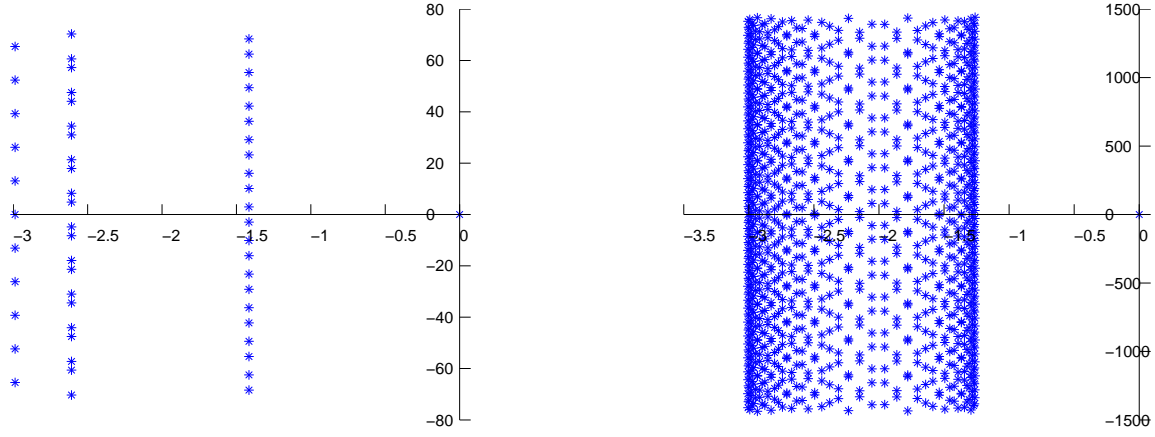


Figure 2: Distribution of eigenvalues for  $a/L = 2/5$  (left), and  $a/L = 41/100$  (right) with  $h_1 = 0.3$ ,  $h_2 = 0.9$ ,  $h_3 = 0.7$ ,  $c = 0.3$  and  $L = 1.8$ . Horizontal lines are the real axes, vertical lines are parallel to the imaginary axes.

in the complex plane. The lines are not necessarily distinct even if the roots are, since the latter may have equal absolute values. In fact, if Eq.(16) has a complex root  $z_k$  then its conjugate  $\bar{z}_k$  is also a root, and the corresponding line contains two different sequences  $s_n^{(k)}$  with arguments of opposite signs. The eigenvalues are spaced twice as densely along such lines, see Fig.2. If more roots have the same absolute value the density increases further, but the eigenvalues remain distinct as long as the roots are.

However, if Eq.(16) does have multiple roots then according to Eq.(17) we get an infinite sequence of eigenvalues with the same multiplicity. Multiple eigenvalues create extra difficulties in computing the Green's function since the eigenmodes are no longer sufficient to express the response (the associated modes also enter). We shall assume henceforth that all the roots of Eq.(16) are simple. Then all the eigenvalues are also simple with the possible exception of  $s = 0$ , which will have multiplicity two if  $\Delta_a(0) = h_1 + h_2 + 2h_3 = 0$ . It corresponds to a bar moving as a rigid body with constant velocity.

When  $p$  is small  $z_k$  can be found in closed form. The simplest such case is  $p = 2$ ,  $q = 1$  so that  $a = \frac{L}{2}$ , i.e. the internal damper is placed exactly in the middle of the interval  $[0, L]$ . We wish to analyze this case in more detail next.

## 5 Damper in the middle

When  $a/L = 1/2$  Eq.(16) reduces to a quadratic equation:

$$Az^2 + Bz + C = 0 \quad \text{with} \quad (18)$$

$$A = (1 + h_1)(1 + h_2)(1 + h_3), \quad B = 2h_3(1 - h_1h_2), \quad C = -(1 - h_1)(1 - h_2)(1 - h_3).$$

As above, we exclude critical cases so that  $A \neq 0$  and  $C \neq 0$ . Then Eq.(18) has two real roots, two complex conjugate roots or one real double root according to whether

$$D := \frac{B^2 - 4AC}{4} = (1 - h_1^2)(1 - h_2^2) + h_3^2(h_1 - h_2)^2 \quad (19)$$

is positive, negative or zero respectively. Moreover, we have explicitly

$$z_{1,2} = \frac{-h_3(1 - h_1h_2) \pm \sqrt{D}}{(1 + h_1)(1 + h_2)(1 + h_3)}. \quad (20)$$

Accordingly, for  $D > 0$  we have two sequences of eigenvalues spaced along two distinct vertical lines, see Eq.(17). They merge into a single line of double eigenvalues for  $D = 0$ , and for  $D < 0$  we have both sequences half-spaced along a single line with

$$\text{Re}(s) = \frac{c}{4L} \ln |z_{1,2}| = \frac{c}{8L} \ln \left| \frac{(1 - h_1)(1 - h_2)(1 - h_3)}{(1 + h_1)(1 + h_2)(1 + h_3)} \right|.$$

Even if  $D \neq 0$  when  $h_1 + h_2 + 2h_3 = 0$  we have a double eigenvalue at zero because of the rigid mode, in which case  $D = (h_1^2 + h_2^2 - 2)^2 \neq 0$ .

As an application, we will determine conditions for 'undamped' behavior of the bar. The idea is this: when  $h_i$ 's are negative they describe active dashpots that add energy to the bar instead of draining it [6]. It is therefore possible that for some combinations of values the same amount of energy is being added as is being drained by the dashpots – the bar behaves as if it were undamped altogether. Clearly, no damping means that all the eigenvalues are purely imaginary or equivalently, that all the roots of Eq.(18) lie on the unit circle since  $z = e^{\frac{sL}{c}}$ .

If both roots are real then  $z = 1, 1$ ,  $z = -1, -1$  or  $z = 1, -1$ . This leads to  $C = A$ ,  $B = \pm 2A$  or  $C = -A$ ,  $B = 0$

respectively. If one of them is complex then its conjugate is also a root and  $z = e^{\pm i\theta}$  with  $\theta \neq 0, \pi$ . We have

$$Az^2 + Bz + C = A(z - e^\theta)(z - e^{-\theta}) = Az^2 - 2A \cos \theta + A,$$

so  $C = A$  and  $\cos \theta = -B/2A$ . The latter condition can be satisfied by a  $\theta \neq 0, \pi$  if and only if  $|B| < 2|A|$ . Two of the real cases above will be subsumed here if we allow  $|B| \leq 2|A|$ .

To make these conditions explicit in terms of  $h_i$ 's let us start with the case  $h_3 = 0$ . This automatically means  $B = 0$ , in particular  $|B| \leq 2|A|$ , so  $C = A$  and  $C = -A$  give us the two available possibilities:

$$(1) \begin{cases} h_3 = 0 \\ h_1 h_2 = -1 \end{cases} \quad \text{and} \quad (2) \begin{cases} h_3 = 0 \\ h_1 + h_2 = 0. \end{cases}$$

In each case  $h_1$  or  $h_2$  can be chosen arbitrarily and the other parameter is uniquely defined. When  $h_3 \neq 0$  we have a similar situation for  $C = -A$ :

$$(3) \begin{cases} h_3 = -\frac{h_1 + h_2}{2} \\ h_1 h_2 = 1. \end{cases}$$

Note that in the second and third case we have  $h_1 + h_2 + 2h_3 = 0$ , which means that there is a double pole at  $s = 0$ . Therefore, we have not just zero damping but rigid motion: the bar will be moving with constant velocity in addition to oscillations.

When  $h_3 \neq 0$  the case of  $C = A$  becomes unexpectedly complicated. After cancellations  $C = A$  reduces to  $h_1 h_2 + h_1 h_3 + h_2 h_3 + 1 = 0$  and we also get

$$\cos \theta = -\frac{1 - h_1^2 h_2^2}{(1 - h_1^2)(1 - h_2^2)}$$

We can solve for  $h_3$  since  $h_1 + h_2 = 0$  leads to one of the above cases, but writing out  $|B| \leq 2|A|$  is not very helpful

for determining  $h_1, h_2$ . We will assume instead that one picks  $h_1$  and  $\cos \theta$  and then solves for  $h_2, h_3$ :

$$(4) \quad \begin{cases} h_3 = -\frac{1+h_1 h_2}{h_1+h_2} \\ h_2 = \pm \sqrt{\frac{(1+\cos \theta)-h_1^2 \cos \theta}{\cos \theta+(1-\cos \theta)h_1^2}}. \end{cases}$$

One can see that the range of parameters in case (4) is two-dimensional in contrast to the first three cases. Still, the choice of  $h_1$  and  $\cos \theta$  is not entirely arbitrary. Aside from the obvious restriction  $-1 \leq \cos \theta \leq 1$  one has to pick  $h_1$  so that the expression under the square root is non-negative. For example, if  $\cos \theta = 1$  then  $h_2 = \pm \sqrt{2-h_1^2}$  and we can only pick  $h_1$  that satisfies  $|h_1| \leq \sqrt{2}$ . In particular,  $|h_1| \leq 1$  always works for  $\cos \theta > 0$ , and  $|h_1| \geq 1$  for  $\cos \theta < 0$ .

For  $C = A$  and  $B = \pm 2A$ , which corresponds to  $\cos \theta = \pm 1$ , we have multiplicity in eigenvalues. Indeed, Eq.(17) shows that all eigenvalues on the imaginary axis are now double zeros of  $\Delta_a(s)$ . Therefore, we have oscillations with linearly increasing amplitudes. Moreover, when  $\cos \theta = 1$  we have that 0 is even a triple pole of  $s\Delta_a(s)$  and hence a triple eigenvalue. Physically, this means that the bar does not just move rigidly, it is accelerating.

## 6 Green's function

By definition, the Green's function  $G(x, \xi, s)$  for system (3)-(4) satisfies

$$\frac{s^2}{c^2} G + 2h_3 \frac{s}{c} \delta(x-a) G - G_{xx}(x, s) = \delta(x-\xi), \quad (21)$$

$$G_x(0, \xi, s) - h_1 \frac{s}{c} G(0, \xi, s) = 0 \quad \text{and} \quad G_x(L, \xi, s) + h_2 \frac{s}{c} G(L, \xi, s) = 0. \quad (22)$$

It can be computed along the same lines as  $\varphi_a, \psi_a$  in section 3, and has different analytic expressions depending on relative positions of  $x, a$  and  $\xi$ . Consider the case  $a < \xi$  first. As a function of  $x$ ,  $G$  satisfies Eq.(3) for  $x < \xi$  and  $x > \xi$ . Therefore, it is equal to  $A\varphi_a(x, s)$  on  $[0, \xi)$  and  $B\psi(x, s)$  on  $(\xi, L]$ . At  $x = \xi$  it is continuous, but has a jump in the first derivative to produce  $\delta(x-\xi)$  in Eq.(21). Namely,  $G_x(\xi^+, \xi, s) - G_x(\xi^-, \xi, s) = -1$  because  $G_{xx}$

enters Eq.(21) with minus. Therefore,  $A, B$  can be found from the system

$$\begin{cases} A\varphi_a(\xi, s) - B\psi(\xi, s) &= 0 \\ A\varphi'_a(\xi, s) - B\psi'(\xi, s) &= 1 \end{cases}$$

Solving for them in the matrix form we get

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{-W[\varphi_a, \psi]} \begin{pmatrix} -\psi' & \psi \\ -\varphi'_a & \varphi_a \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

But for  $\xi > a$  we have from Eq.(11) that  $\psi_a(\xi, s) = \psi(\xi, s)$  so that  $W[\varphi_a, \psi_a] = W[\varphi_a, \psi] = -\frac{c}{s}\Delta_a(s)$  from Eq.(12).

Therefore,

$$A = c \frac{\psi(\xi, s)}{s\Delta_a(s)} \quad \text{and} \quad B = c \frac{\varphi_a(x, s)}{s\Delta_a(s)},$$

so that

$$G_{a<\xi}(x, \xi, s) = \frac{c}{s\Delta_a(s)} \begin{cases} \varphi_a(x, s)\psi(\xi, s), & x < \xi \\ \psi(x, s)\varphi_a(\xi, s), & x > \xi \end{cases} \quad (23)$$

Analogously, for  $a > \xi$  in the latter case we have

$$G_{a>\xi}(x, \xi, s) = \frac{c}{s\Delta_a(s)} \begin{cases} \varphi(x, s)\psi_a(\xi, s), & x < \xi \\ \psi_a(x, s)\varphi(\xi, s), & x > \xi \end{cases} \quad (24)$$

It will be convenient for us to rewrite  $G$  in a form that is both more explicit, and makes its symmetry  $G(x, \xi, s) =$

$G(\xi, x, s)$  manifest. To this end, we introduce

$$g_{\varphi\psi}(x, \xi, s) := \begin{cases} \varphi(x, s)\psi(\xi, s), & x < \xi \\ \varphi(\xi, s)\psi(x, s), & x > \xi \end{cases},$$

and compute

$$g_{\varphi\psi}(x, \xi, s) = \frac{1}{4} \left[ (1+h_1)(1+h_2)e^{\frac{s(L-|x-\xi|)}{c}} + (1-h_1)(1+h_2)e^{\frac{s(L-(x+\xi))}{c}} \right. \\ \left. + (1+h_1)(1-h_2)e^{-\frac{s(L-(x+\xi))}{c}} + (1-h_1)(1-h_2)e^{-\frac{s(L-|x-\xi|)}{c}} \right]. \quad (25)$$

Analogously, let

$$g_{s\psi}(x, \xi, s) := \begin{cases} \sinh \frac{s(x-a)}{c} \psi(\xi, s), & x < \xi \\ \sinh \frac{s(\xi-a)}{c} \psi(x, s), & x > \xi \end{cases} \quad \text{and} \quad g_{s\varphi}(x, \xi, s) := \begin{cases} \sinh \frac{s(a-\xi)}{c} \varphi(x, s), & x < \xi \\ \sinh \frac{s(a-x)}{c} \varphi(\xi, s), & x > \xi. \end{cases}$$

Then

$$g_{s\psi}(x, \xi, s) = \frac{1}{4} \left[ (1+h_2)e^{\frac{s(L-a-|x-\xi|)}{c}} - (1+h_2)e^{\frac{s(L+a-(x+\xi))}{c}} \right. \\ \left. + (1-h_2)e^{-\frac{s(L+a-(x+\xi))}{c}} - (1-h_2)e^{-\frac{s(L-a-|x-\xi|)}{c}} \right] \quad (26)$$

$$g_{s\varphi}(x, \xi, s) = \frac{1}{4} \left[ (1+h_1)e^{\frac{s(a-|x-\xi|)}{c}} - (1+h_1)e^{\frac{s((x+\xi)-a)}{c}} + (1-h_1)e^{-\frac{s((x+\xi)-a)}{c}} - (1-h_1)e^{-\frac{s(a-|x-\xi|)}{c}} \right]. \quad (27)$$

Since the  $g_{\varphi\psi}$  part is common to all arrangements of  $x$ ,  $a$  and  $\xi$  we get

$$G(x, \xi, s) = \frac{c}{s\Delta_a(s)} \left[ g_{\varphi\psi}(x, \xi, s) + 2h_3 H(x-a)H(\xi-a) \varphi(a, s) g_{s\psi}(x, \xi, s) \right. \\ \left. + 2h_3 H(a-x)H(a-\xi) \psi(a, s) g_{s\varphi}(x, \xi, s) \right]. \quad (28)$$

Note that the last two terms are non-zero only when  $x$  and  $\xi$  are on the same side of  $a$ . Therefore, whenever  $a$  separates  $x$  and  $\xi$  the Green's function reduces to the first term.

As mentioned earlier, to solve the original problem we need to invert the inverse Laplace transform  $\Gamma(x, \xi, t) := \mathcal{L}^{-1}[G(x, \xi, s)]$ , but  $G$  is too complicated to allow inversion in a closed form. We are forced to expand it into a series over functions with simpler dependence on  $s$  and invert it termwise. In the spectral method, which we follow

here, these functions are the partial fractions  $\frac{1}{(s-p)^m}$  and  $\mathcal{L}^{-1}\left[\frac{1}{(s-p)^m}\right] = \frac{t^{m-1}}{(m-1)!} e^{pt}$ , where  $p$  are the poles of  $G$ . Since the denominator of  $G$  is  $s\Delta_a(s)$  its poles are exactly the eigenvalues  $s_n^{(k)}$  from Eq.(17) and 0 (none of them cancel with the numerator).

Let us assume for the moment that expansion into partial fractions is possible and moreover, that all the poles except possibly  $s = 0$  are simple. A general theorem on Green's functions [10, chap 1, sect 3] implies that

$$G(x, \xi, s) = G_0(x, \xi, s) + \sum_{s_n^{(k)} \neq 0} \frac{\varphi_a(x, s_n^{(k)}) \varphi_a(\xi, s_n^{(k)})}{A_n^{(k)}(s - s_n^{(k)})}, \quad (29)$$

where  $G_0$  is the principal part at  $s = 0$ ,  $\varphi_a(x, s_n^{(k)})$  are the eigenmodes corresponding to  $s_n^{(k)}$ , and  $A_n^{(k)}$  are numerical coefficients. As shown in the Appendix, in our case the latter can be computed explicitly:

$$A_n^{(k)} = \frac{2s_n^{(k)}}{c^2} \int_0^L \varphi_a(\xi, s_n^{(k)})^2 d\xi + \frac{h_1}{c} \varphi_a(0, s_n^{(k)})^2 + \frac{h_2}{c} \varphi_a(L, s_n^{(k)})^2 + \frac{2h_3}{c} \varphi_a(a, s_n^{(k)})^2. \quad (30)$$

If  $s = 0$  is also a simple pole, i.e.  $\Delta_a(0) \neq 0$ , then  $G_0(x, \xi, s) = cF(x, \xi, 0)/s\Delta_a(0)$ , where  $F$  is the bracketed expression in Eq.(28). By inspection, from Eqs.(6),(7),(10),(11) we see that  $\varphi = \psi = \varphi_a = \psi_a = 1$  for  $s = 0$ . Recall also that  $\Delta_a(0) = h_1 + h_2 + 2h_3$ . Therefore,  $F(x, \xi, 0) = 1$  and

$$G_0(x, \xi, s) = \frac{c}{(h_1 + h_2 + 2h_3)s} \quad (31)$$

The Laplace transform is now easily inverted and

$$\Gamma(x, \xi, t) := \mathcal{L}^{-1}[G(x, \xi, s)] = \frac{c}{h_1 + h_2 + 2h_3} + \sum_{k=1}^p \sum_{n=-\infty}^{\infty} \frac{1}{A_n^{(k)}} \varphi_a(x, s_n^{(k)}) \varphi_a(\xi, s_n^{(k)}) e^{s_n^{(k)} t} \quad (32)$$

If  $s = 0$  is a double eigenvalue, i.e. a simple zero of  $\Delta_a$ , the answer is more cumbersome. For example, when  $h_3 = -\frac{h_1+h_2}{2}$  and  $h_2 = \frac{1}{h_1}$  all the eigenvalues are on the imaginary axis and there is a double pole at zero. The principal part of the Green's function at  $s = 0$  is provided with  $G_0(x, \xi, s) = \frac{c_1}{s} + \frac{c_2}{s^2}$  where  $c_1$  and  $c_2$  are computed



in the usual way [11, chap 5, sect 5-11]. If, for convenience, we set

$$H_a(x, \xi) := H(x - a)H(\xi - a) + H(a - x)H(a - \xi) = \begin{cases} 1, & a \text{ is on the same side of } x \text{ and } \xi \\ 0, & a \text{ separates } x \text{ and } \xi \end{cases}, \quad (33)$$

then  $c_1$  and  $c_2$  are provided as

$$c_1 = \frac{h_1 c}{2(L(h_1^2 - 1) - a(h_1^4 - 1))} \left[ (h_1^2 + 1)(|x - \xi| - (x + \xi) + 2a) H_a(x, \xi) \right. \\ \left. - (h_1^2 + 1)|x - \xi| + (h_1^2 - 1)(x + \xi) + 2L \right], \quad (34)$$

$$c_2 = \frac{h_1^2 c^2}{L(h_1^2 - 1) - a(h_1^4 - 1)}. \quad (35)$$

The corresponding terms in  $\Gamma(x, \xi, t)$  are  $c_1 + c_2 t$ . Physically, the bar is moving as a rigid body with constant velocity in addition to vibrating. When  $s = 0$  has higher multiplicity the bar will be accelerating. In the case when infinitely many eigenvalues are multiple, the simple template Eq.(29) for the partial fraction expansion no longer applies. We investigate when such expansion is possible at all next.

## 7 Critical cases

Through the previous section we assumed that the Green's function can be expanded into partial fractions over its poles. This is not always the case as observed already in [6]. In general, if one takes a function like  $e^{-s}$  with no poles at all, it will have no expansion in terms of partial fractions. The usual way of proving convergence is to apply the Cauchy residue theorem to circles of increasing radii. But for this argument to work, the contour integrals over the circles must tend to zero as the radii tend to infinity. This is violated for functions like  $e^{-s}$ . To ensure the desired convergence it suffices, for example, that  $|G(x, \xi, s)| \leq \text{const}/|s|$  outside disks of any fixed size surrounding the poles of  $G$  (the constant will depend on the size chosen). When this inequality does not hold we call the case critical.

According to Eq.(28),  $G(x, \xi, s) = \frac{cF(s)}{s\Delta_a(s)}$  with  $F(s)$  being the expression in brackets. Since  $1/s$  factor is already present it would suffice that  $F(s)/\Delta_a(s)$  be bounded away from the poles. A quick look at explicit expressions for

$F$  and  $\Delta_a$  shows both to be exponential sums of the type  $a_1 e^{\alpha_1 s} + \dots + a_m e^{\alpha_m s}$  with real exponents  $\alpha_i$ . Quite a bit is known about such sums, e.g. their zeros are located within a vertical strip  $|\operatorname{Re}(s)| \leq \omega$ , and within this strip the sum is bounded. Moreover, on the complement to all disks of any fixed size surrounding the zeros exponential sums are uniformly separated from 0 [9]. Consequently, we only need to worry about  $F(s)/\Delta_a(s)$  being bounded when  $\operatorname{Re}(s) \rightarrow \pm\infty$ .

Clearly, for  $\operatorname{Re}(s) \rightarrow \infty$  ( $-\infty$ ) the term with the largest (smallest)  $\alpha_i$  dominates an exponential sum. We conclude that  $F/\Delta_a$  is uniformly bounded away from the poles if and only if the largest (smallest) exponent in the denominator is greater (less) than the largest (smallest) exponent in the numerator. To put it differently: for boundedness all exponents in the numerator must lie between the largest and the smallest ones in the denominator. Let us take count of these exponents in Eq.(28) for  $\xi > a$ , the other case is analogous. We find that in the numerator the following terms occur

$$e^{\frac{s}{c}(L-|x-\xi|)}, \quad e^{\frac{s}{c}(L-(x+\xi))}, \quad e^{-\frac{s}{c}(L-|x-\xi|)}, \quad e^{-\frac{s}{c}(L-(x+\xi))} \quad (36)$$

$$e^{\frac{s}{c}(L-2a-|x-\xi|)}, \quad e^{\frac{s}{c}(L+2a-(x+\xi))}, e^{-\frac{s}{c}(L-2a-|x-\xi|)}, \quad e^{-\frac{s}{c}(L+2a-(x+\xi))}, \quad (37)$$

and the terms in the second row are multiplied by 0 when  $x < a$ . On the other hand,  $\Delta_a$  contains

$$e^{\frac{s}{c}L}, \quad e^{\frac{s}{c}(L-2a)}, \quad e^{-\frac{s}{c}L}, \quad e^{-\frac{s}{c}(L-2a)}. \quad (38)$$

We see that the exponents of Eq.(36) do lie between  $-L$  and  $L$ , so the condition for boundedness is satisfied assuming that coefficients in front of  $e^{\frac{s}{c}L}$  and  $e^{-\frac{s}{c}L}$  in  $\Delta_a(s)$  are non-zero. By Eq.(14) this means that  $(1+h_1)(1+h_2)(1+h_3) \neq 0$  and  $(1-h_1)(1-h_2)(1-h_3) \neq 0$ , or equivalently  $h_i \neq \pm 1$  for  $i = 1, 2, 3$ . For ODE of any order with linear boundary conditions for non-criticality are derived in [2].

When say  $h_2$  is 1, the only surviving exponents in Eq.(38) are the first and the third. This changes the calculation since some of the exponents in the numerator, like  $L - (x + \xi)$ , are perfectly capable of being less than  $L - 2a$ . One may hope that 'bad' terms in the numerator also vanish, but no, for  $h_2 = 1$  the first two exponents from each row of Eq.(36) have non-zero coefficients.

When  $h_i = \pm 1$  for at least one  $i$  the spectral method no longer applies and one has to look for other ways to invert

the Laplace transform. Before discussing them let us explain physical reasons for critical behavior. Take  $h_2 = 1$  and note that in the original initial-boundary problem the right boundary condition becomes  $u_t(L, t) + c u_x(L, t) = 0$ , while the equation away from the internal damper is  $u_{tt} - c^2 u_{xx} = 0$ . It is well-known that its solution splits into left and right traveling waves which satisfy  $u_t - c u_x = 0$  and  $u_t + c u_x = 0$  respectively. The left traveling waves play no role near the right boundary (there is nowhere for them to come from), while the right traveling waves satisfy the boundary condition automatically. In other words, the right boundary condition has no effect whatsoever, and we effectively get a problem on the semi-infinite interval  $[0, \infty)$ , but with initial data restricted to  $[0, L]$ . Physically, the right boundary becomes transparent allowing the waves to pass through it without any reflection back. If in addition there is no internal damper ( $h_3 = 0$ ) then standing waves, a.k.a. eigenmodes, can not form, so of course one can not expand over them. When the internal damper is present, it reflects some of the waves at  $a$  forming standing waves on  $[0, a]$ , but not on  $[a, L]$ . The expansion is still impossible even though some eigenmodes are present. More physical analysis of transparent boundaries is given in [6].

When  $h_2 = -1$  the situation can be analyzed similarly. Now the boundary condition is  $u_t(L, t) - c u_x(L, t) = 0$  while the equation prescribes  $u_t + c u_x = 0$ . This is only possible if  $u_t(L, t) = u_x(L, t) = 0$ , conditions impossible to satisfy in general. If the initial disturbance is localized away from the right boundary the solution will exist only up to a time necessary for the right traveling waves to reach it. As  $h_2$  approaches  $-1$  the reflection coefficient at the right boundary grows without bound [6], so the physical reason for the non-existence is that reflection would involve an infinite energy transfer. One can not say however, that solution blows up in finite time since it behaves 'normally' before the boundary is reached and ceases to exist at that instant.

How does one invert the Laplace transform in critical cases? Suppose first that there is no internal damper ( $h_3 = 0$ ) and the right boundary is transparent ( $h_2 = 1$ ). Then there is only one exponent left in the denominator of the Green's function  $\Delta_a(s) = (1 + h_1)e^{\frac{sL}{c}}$ . This exponent can be combined with the ones in the numerator so that the Green's function reduces to a linear combination of exponents divided by  $s$

$$G(x, \xi, s) = \frac{c}{2s} \left[ e^{-\frac{s}{c}|x-\xi|} + \frac{1-h_1}{1+h_1} e^{-\frac{s}{c}(x+\xi)} \right]. \quad (39)$$

Since  $\mathcal{L}^{-1}[\frac{1}{s}e^{-\alpha s}] = H(t - \alpha)$ , where  $H$  is the Heaviside function,  $\Gamma(x, \xi, t) := \mathcal{L}^{-1}[G(x, \xi, s)]$  can even be found in

closed form:

$$\Gamma(x, \xi, t) = \frac{c}{2} \left[ H(ct - |x - \xi|) + \frac{1 - h_1}{1 + h_1} H(ct - (x + \xi)) \right]. \quad (40)$$

Physically, we get left and right traveling waves with the former reflected from the left boundary just once. Thus, it is not surprising that the solution is a finite superposition of traveling waves, just as it is for the wave equation on the entire line according to the d'Alembert's solution.

Summarizing, for  $h_3 = 0$  we have a simple dichotomy: either both boundaries are non-transparent and the solution can be found by the spectral method, or one or both are and the solution can be found in a closed form as a finite superposition of traveling waves. When  $h_3 \neq 0$  this dichotomy fails because standing waves may be able to form on part of the interval between one of the boundaries and the internal damper. Then one is forced to either combine traveling and standing waves, or to use an infinite number of traveling waves. We shall not treat such intermediate cases in this paper.

However, if both boundaries are transparent ( $h_1 = h_2 = 1$ ) the standing waves can not form at all and one can find a closed form solution again. Then from Eq.(28)

$$G(x, \xi, s) = \frac{c}{2(1 + h_3)s} \left[ e^{-\frac{s}{c}|x - \xi|} + h_3 H_a(x, \xi) \left( e^{-\frac{s}{c}|x - \xi|} - e^{-\frac{s}{c}|x + \xi - 2a|} \right) \right]; \quad (41)$$

$$\Gamma(x, \xi, t) = \frac{c}{2(1 + h_3)} \left[ H(ct - |x - \xi|) + h_3 H_a(x, \xi) \left( H(ct - |x - \xi|) - H(ct - |x + \xi - 2a|) \right) \right]. \quad (42)$$

## 8 Vibratory response

When computing the Green's function we set all the initial-boundary data to zero and it is now time to bring it back. The Laplace transform of the original system (1),(2) is

$$\frac{s^2}{c^2} U(x, s) + 2h_3 \frac{s}{c} \delta(x - a) U(x, s) - U_{xx}(x, s) = s \frac{u(x, 0)}{c^2} + \frac{\dot{u}(x, 0) + p(x, s)}{c^2} + 2\frac{h_3}{c} \delta(x - a) u(x, 0), \quad (43)$$

$$U_x(0, s) - h_1 \frac{s}{c} U(0, s) = -\frac{h_1}{c} u(0, 0) \quad \text{and} \quad U_x(L, s) + h_2 \frac{s}{c} U(L, s) = \frac{h_2}{c} u(L, 0). \quad (44)$$

If not for the inhomogeneity in the boundary conditions we could have solved Eqs.(43),(44) by convolution with the Green's function. Let us denote this convolution  $\tilde{U}(x, s)$ , i.e.

$$\tilde{U}(x, s) = \int_0^L s G(x, \xi, s) \frac{u(\xi, 0)}{c^2} d\xi + \int_0^L G(x, \xi, s) \frac{\dot{u}(\xi, 0) + p(\xi, s)}{c^2} d\xi + 2 \frac{h_3}{c} G(x, a, s) u(a, 0). \quad (45)$$

As shown in Appendix, inhomogeneity in the boundary conditions introduces two extra terms to the right hand side of Eq.(45) analogous to its last term, so

$$U(x, s) = \tilde{U}(x, s) + \frac{h_1}{c} G(x, 0, s) u(0, 0) + \frac{h_2}{c} G(x, L, s) u(L, 0). \quad (46)$$

Since  $\Gamma(x, \xi, t) := \mathcal{L}^{-1}[G(x, \xi, s)]$  properties of Laplace transform immediately imply

$$\begin{aligned} \mathcal{L}^{-1}[s G(x, \xi, s)] &= \Gamma_t(x, \xi, t) \\ \mathcal{L}^{-1}[G(x, \xi, s) p(\xi, s)] &= \int_0^t \Gamma(x, \xi, t - \tau) p(\xi, \tau) d\tau. \end{aligned}$$

Technically speaking,  $\mathcal{L}^{-1}[s G(x, \xi, s)] = \Gamma_t(x, \xi, t) + \Gamma(x, \xi, 0) \delta(t)$ , but we have ignored the delta function since it does not contribute for  $t > 0$ . Therefore, solution to the initial-boundary problem Eqs.(1),(2) can be written in the form

$$\begin{aligned} u(x, t) &= \frac{1}{c} \left[ h_1 u(0, 0) \Gamma(x, 0, t) + h_2 u(L, 0) \Gamma(x, L, t) + 2h_3 u(a, 0) \Gamma(x, a, t) \right] \\ &+ \frac{1}{c^2} \int_0^L \left[ \Gamma_t(x, \xi, t) u(\xi, 0) + \Gamma(x, \xi, t) \dot{u}(\xi, 0) \right] d\xi + \frac{1}{c^2} \int_0^t \int_0^L \Gamma(x, \xi, t - \tau) p(\xi, \tau) d\xi d\tau \end{aligned} \quad (47)$$

If  $h_i \neq \pm 1$ ,  $h_1 + h_2 + 2h_3 \neq 0$  and all roots of algebraic equation Eq.(16) are simple then all our assumptions for the validity of Eq.(32) are satisfied and we can substitute it into Eq.(47). Taking into account that

$$\begin{aligned} \Gamma(x, 0, t) &= \frac{c}{h_1 + h_2 + 2h_3} + \sum_{k=1}^p \sum_{n=-\infty}^{\infty} \frac{1}{A_n^{(k)}} \varphi_a(x, s_n^{(k)}) e^{s_n^{(k)} t} \\ \Gamma_t(x, \xi, t) &= \sum_{k=1}^p \sum_{n=-\infty}^{\infty} \frac{s_n^{(k)}}{A_n^{(k)}} \varphi_a(x, s_n^{(k)}) \varphi_a(\xi, s_n^{(k)}) e^{s_n^{(k)} t}, \end{aligned}$$

we have for the vibratory response:

$$\begin{aligned}
u(x, t) = & \frac{1}{h_1 + h_2 + 2h_3} \left[ h_1 u(0, 0) + h_2 u(L, 0) + 2h_3 u(a, 0) + \frac{1}{c} \int_0^L \left[ \dot{u}(\xi, 0) + \int_0^t p(\xi, \tau) d\tau \right] d\xi \right] \\
& + \sum_{k=1}^p \sum_{n=-\infty}^{\infty} \frac{\varphi_a(x, s_n^{(k)}) e^{s_n^{(k)} t}}{c^2 A_n^{(k)}} \left[ \left[ c h_1 u(0, 0) + c h_2 u(L, 0) \varphi_a(L, s_n^{(k)}) + 2c h_3 u(a, 0) \varphi_a(a, s_n^{(k)}) \right] \right. \\
& \left. + \int_0^L \left[ s_n^{(k)} u(\xi, 0) + \dot{u}(\xi, 0) + \int_0^t p(\xi, \tau) e^{-s_n^{(k)} \tau} d\tau \right] \varphi_a(\xi, s_n^{(k)}) d\xi \right]. \quad (48)
\end{aligned}$$

Formula Eq.(47) remains valid of course even if the eigenmode expansion is inapplicable. For example, in those critical cases when  $\Gamma(x, \xi, t)$  is found in closed form, see Eqs.(40),(42), we can use it to find the vibratory response as well. When substituting into Eq.(47) one should keep in mind that  $H_t(t - \alpha) = \delta(t - \alpha)$ , where  $\delta(t - \alpha)$  is the delta function of Dirac. Its convolution with any function simply returns the function's value at  $\alpha$ . Also, when terms like  $H(\xi - a)H(ct - |x - \xi|)$  are integrated over  $\xi$  from 0 to  $L$  one has to consider cases  $x > \xi$  and  $x < \xi$  separately to remove the absolute value, e.g.

$$\int_0^L H(\xi - a)H(ct - |x - \xi|) w(\xi) d\xi = \int_{\max\{a, x-ct\}}^x w(\xi) d\xi + \int_{\max\{a, x\}}^{\min\{L, x+ct\}} w(\xi) d\xi$$

Note that the first integral is assumed to vanish if its lower limit exceeds its upper limit, e.g.  $a > x$ . Explicit expressions are cumbersome and are given in the Appendix.

## 9 Numerical issues

In this section we illustrate the behavior of the system for various values of parameters  $h_1$ ,  $h_2$  and  $h_3$ . The critical cases are of particular interest as they demonstrate somewhat counter-intuitive behavior of the bar. Our analytical expressions are calculated using Maple and in some cases we compare results to a MATLAB finite element implementation. We subject the bar to initial displacement only in order to illustrate how each damper at the boundary and along the bar modifies traveling waves. We use a Gaussian function  $0.1 \exp(-(x-\mu)^2/(2\sigma^2))/(\sigma\sqrt{2\pi})$  as an approximation of an impulse function where  $\sigma = 0.1$ . We first begin with some special values of the parameters  $h_i$ . For all figures  $c = 1.5$  and  $L = 1.8$ .

### 9.1 Response of the system for $h_2 = 1, h_3 = 0$

Figure 3 shows the response of the system when  $h_1 = 0.5, h_2 = 1$  and  $h_3 = 0$ . The right boundary is transparent and the left-half of the traveling wave disappears on the right while the right-half get reflected from the left boundary and then it too disappears on the right. This behavior can be termed super-stable because the bar comes to rest in finite time. Such a phenomenon can only be appropriately observed using explicit solutions and it is the consequence of the continuum nature of the system. Once we discretize the system this behavior disappears and the bar could only come to rest at infinite time. The reason for this is an exponential nature of the solution for a discrete system.

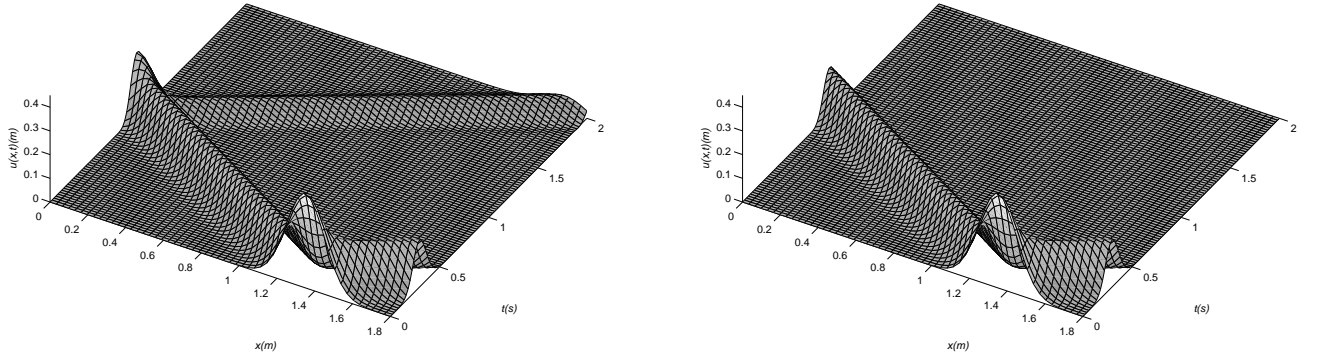


Figure 3: Vibratory response for  $h_3 = 0, h_1 = 0.5, h_2 = 1$  (left), and  $h_1 = h_2 = 1$  (right).

### 9.2 Response of the system for $h_1 = h_2 = 1, h_3 \neq 0$

Figure 4 depicts the response of the system for  $h_1 = h_2 = 1, h_3 = .5$ . The internal damper is placed at distance  $a = 0.5$  from the left end of the bar. The initial pulse consists of two Gaussian functions centered at  $0.25 L$  and  $0.75 L$ . We see that a wave gets partially reflected by the internal damper. This can be best observed in the Figure 2 for the left-half of the right impulse. For greater values of the  $h_3$  parameter the internal damper acts almost as if it is the fixed point of the bar and waves get reflected to a greater extent while only a small portion of the wave is passes through.

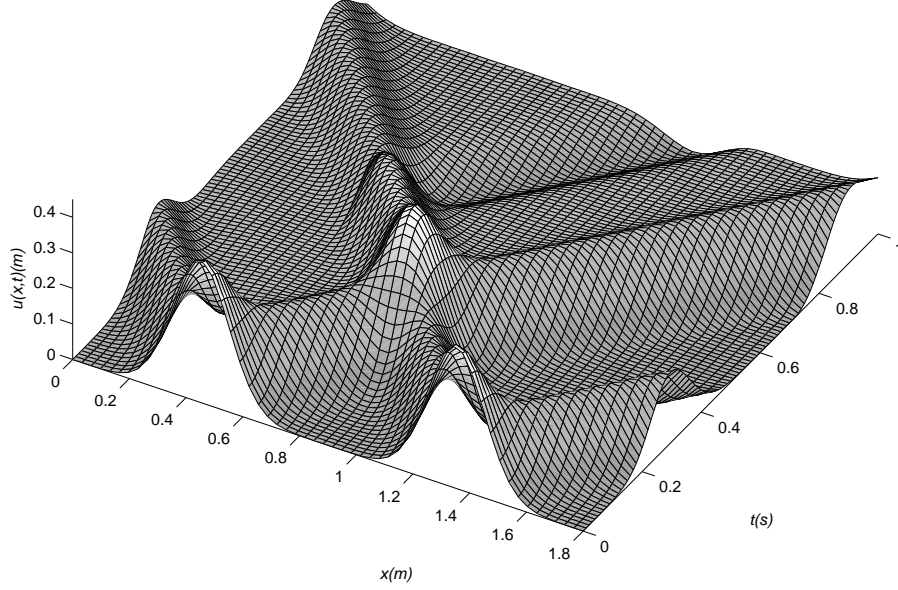


Figure 4: Vibratory response for  $h_1 = h_2 = 1$  and  $h_3 = 0.5$

### 9.3 Response of the system for $h_2 = 0.99$ , $h_3 \neq 0$

Figure 5 depicts the response of the system for  $h_1 = 0.3$ ,  $h_2 = 0.99$  and  $h_3 = 0.7$  using 40 eigenfunctions and a Gaussian impulse at  $\mu = 0.25 L$ . The internal damper is placed at distance  $a = 0.5 L$  from the left end of the bar. In this case the right boundary is almost transparent which precludes formation of standing waves on the right of the internal damper. The left boundary, however, is reflective and the standing waves are clearly visible on Figure 5. This solution was validated with a finite element implementation (FEM) calculated with 160 elements and the error was found to be within 0.001. As the number of elements in FEM was increased this difference became smaller indicating the efficiency of the analytical expression.



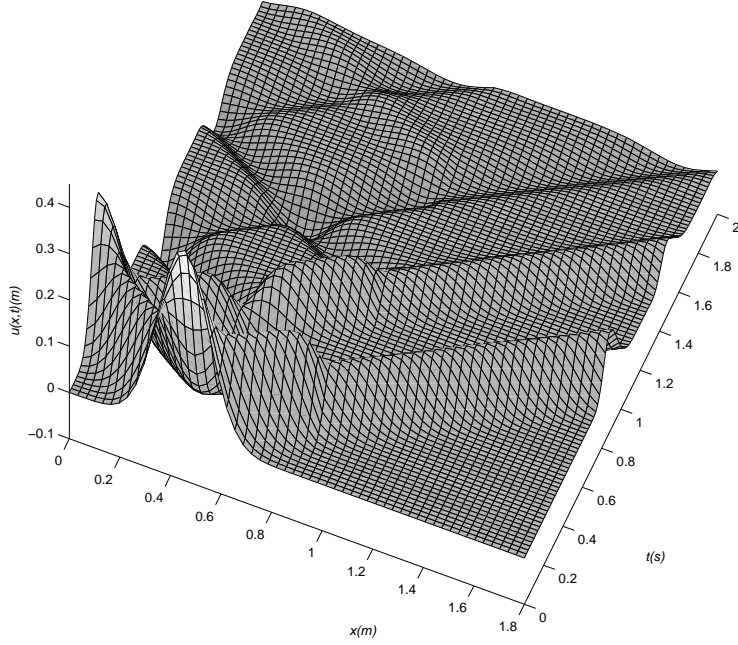


Figure 5: Vibratory response for  $h_1 = 0.3$ ,  $h_2 = 0.99$  and  $h_3 = 0.7$ .

#### 9.4 Response of the system for $h_1 h_2 = 1$ and $h_3 = -\frac{h_1 + h_2}{2}$

Figure 6 shows the response of the system for  $h_1 = 3/10$ ,  $h_2 = 10/3$ ,  $h_3 = -109/60$  and  $a = L/2$  with a Gaussian impulse at  $\mu = 0.25 L$ . This is the case 3) from section 5. In this case the system has a double pole at zero and all the eigenvalues are imaginary, i.e. there is no damping present in the bar. The amount of energy lost at the left and right boundaries is returned into the bar by the damper in the middle. Therefore, the displacement of every point on the bar undergoes periodic motion as depicted in Figure 6. Furthermore, this case is significant since FEM does not yield the correct result. Our simulations indicate that the internal damper greatly decreases the accuracy of FEM eigenvalue computation. As a result, when all the poles are on the imaginary axis errors in the real part significantly distort the response. Similar situation occurs in case 4) of section 5. However, in both cases FEM response is even more distorted by spurious eigenvalues which we discuss next.

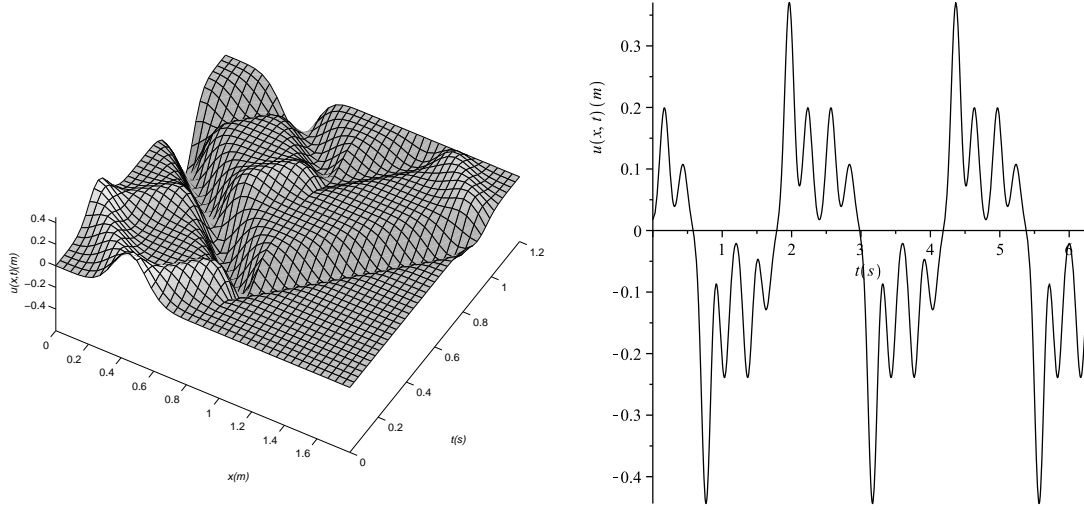


Figure 6:  $u(x, t)$  for  $h_1 = 3/10$ ,  $h_2 = 10/3$ ,  $h_3 = -109/60$  and  $a = 0.5 L$ (left) and  $u(\frac{1}{3}L, t)$  (right).

## 9.5 Spurious eigenvalues in FEM

We have encountered a phenomenon for which we do not have an explanation at this time. We have observed that for some values of parameters  $h_i$  the stable continuous system becomes unstable when discretized by the FEM method. One set of parameters that produces such a behavior is  $h_1 = 0.7$ ,  $h_2 = -1.5$  and  $h_3 = 0$  for which the distribution of its eigenvalues is depicted in Figure 7. It is clear that the continuous system is stable since there are no eigenvalues with positive real parts.

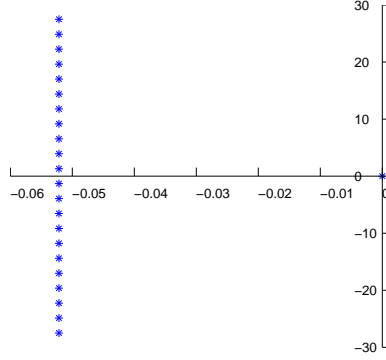


Figure 7: Eigenvalues in the complex plane for  $h_1 = 0.7$ ,  $h_2 = -1.5$ .

If, however, one discretizes the system an eigenvalue with a positive real part will arise. It can be observed that a spurious eigenvalue lies on the positive part of the real axis in the complex plane which forces the discrete system to become unstable. The situation is not improved by increasing the number of finite elements. On the contrary, the real part of the spurious eigenvalue will increase to infinity making the system even more unstable. This is counter-intuitive since we know that the continuous system is equivalent to the discrete one as the number of elements tends to infinity. Note that this phenomenon has nothing to do with the system being unconstrained. Therefore, it can be concluded that at least some stability regions for some parameters of the continuous system become so distorted that after discretization we observe unstable behavior. This phenomenon may have its roots in the non-self-adjointness of the continuous system and at this time it is unclear to us how a discretization of such a system changes its behavioral pattern.

## 10 Conclusions

We studied longitudinal vibrations of a bar with viscous ends and internal damper. The corresponding eigenvalue problem contains the spectral variable in the boundary conditions and has complex-valued, non-orthogonal eigenmodes. Behavior of the system is controlled by four dimensionless parameters, the three damping coefficients  $h_i$ , and the relative position of the internal damper  $a/L$ . Our main observations are summarized below.

Despite the unconventional nature of the eigenvalue problem the eigenmodes can be found explicitly if the

eigenvalues are known. When there is no internal damper or it is located in the middle of the bar the eigenvalues can be found analytically as well. Otherwise, for rational  $a/L$  their determination reduces to solving an algebraic equation. Distribution of eigenvalues is hypersensitive to the value of  $a/L$ : its complexity grows swiftly with the denominator of  $a/L$ , and it becomes pseudo-random for irrational  $a/L$ .

When the values of  $h_i$  are not restricted to be non-negative, i.e. the dampers are allowed to be boosters, effective zero damping may occur. Combinations of parameters that lead to zero damping are non-trivial, but can be found analytically at least when there is no internal damper or it is located in the middle of the bar.

Generically, the eigenmodes are sufficient to expand the Green's function. An explicit series solution can then be found for the vibratory response, it is a superposition of exponentially damped (or boosted) standing waves. These waves however, are complex-valued. Non-generic behavior occurs in two different situations. The first one is that the eigenvalues may have higher multiplicity and the associated modes are required for complete expansion. This only occurs when Eq.(16) has multiple roots or  $h_1 + h_2 + 2h_3 = 0$ . In practice, one can sidestep the issue by slightly perturbing the damping coefficients to resolve the multiplicity.

The second situation is critical behavior, when the eigenvalues disappear partly or fully. This only happens for  $h_i = \pm 1$ , but perturbing the coefficients will lead to a qualitatively different picture at large times. When  $h_i = 1$  the corresponding damper turns into a perfect absorber draining all the energy from the bar in finite time. With no reflection the standing waves either can not form at all (super-stability), or can only form on a part of the bar accounting for scarcity of the eigenmodes. When  $h_i = -1$  one of the dampers turns into an infinite amplifier and the solution can only exist before the traveling waves reach that damper (super-instability). Mathematically, the eigenmode expansion is impossible because the Laplace transformed Green's function is unbounded at infinity and is not equal to the sum of partial fractions over its poles. When no eigenmodes exist the solution can be found in closed form, but partial cases like  $h_1 \neq 1, h_2 = 1, h_3 \neq 0$  require further work. Neither eigenmode expansion, nor closed form solution are possible for such cases. This can only happen when the internal damper is present.

FEM provides good approximation of the vibratory response for small times even in critical cases. However, it is unreliable for determination of eigenvalues even when the number of elements is large. Moreover, it universally produces eigenvalues with large real parts that do not converge to any actual eigenvalues as the number of elements is increased. When these spurious eigenvalues have positive real parts the FEM vibratory response is also unreliable for large times.

## Appendix A: Coefficients of partial fractions

In this appendix we derive an explicit formula for the coefficients of the partial function expansion of the Green's function. For simplicity, within this Appendix we write  $s_n$ ,  $A_n$  instead of  $s_n^{(k)}$ ,  $A_n^{(k)}$ .

To this end consider the integral

$$I := \int_0^L G(x, \xi, s) \left( \frac{s^2}{c^2} \varphi_a(\xi, s_n) - 2h_3 \frac{s}{c} \delta(\xi - a) \varphi_a(\xi, s_n) - \varphi_a''(\xi, s_n) \right) d\xi. \quad (\text{A.1})$$

Taking into account that  $G(\xi, x, s)$  and  $\varphi_a(\xi, s_n)$  both satisfy the boundary conditions the integral  $I$  becomes after integration by parts,

$$I = \varphi_a(x, s_n) + \frac{s - s_n}{c} [h_1 G(x, 0, s) \varphi_a(0, s_n) + h_2 G(x, L, s) \varphi_a(L, s_n)]. \quad (\text{A.2})$$

Now we wish to take the limit  $s \rightarrow s_n$ . Contrary to intuition, the second term does not tend to 0 when  $s \rightarrow s_n$  because  $G$  has a pole at  $s_n$ . Indeed, we have from Eq.(29) that  $G(x, \xi, s) = \frac{\varphi_a(x, s_n) \varphi_a(\xi, s_n)}{A_n(s - s_n)} + \text{Reg}(x, \xi, s)$  where  $\text{Reg}(x, \xi, s)$  is regular at  $s = s_n$ . Substituting this expression we find that

$$\begin{aligned} I &= \varphi_a(x, s_n) + \frac{1}{c} \left[ h_1 \frac{\varphi_a(x, s_n) \varphi_a(0, s_n)^2}{A_n} + h_2 \frac{\varphi_a(x, s_n) \varphi_a(L, s_n)^2}{A_n} \right] + O(s - s_n) \\ &\xrightarrow{s \rightarrow s_n} \varphi_a(x, s_n) \left[ 1 + \frac{1}{A_n c} (h_1 \varphi_a(0, s_n)^2 + h_2 \varphi_a(L, s_n)^2) \right] \end{aligned} \quad (\text{A.3})$$

On the other hand, we can compute  $I$  in a different way. Since  $\varphi_a''(\xi, s_n) = 2h_3 \frac{s_n}{c} \delta(\xi - a) \varphi_a(\xi, s_n) + \frac{s_n^2}{c^2} \varphi_a(\xi, s_n)$  from the equation we have

$$2h_3 \frac{s}{c} \delta(\xi - a) \varphi_a(\xi, s_n) + \frac{s^2}{c^2} \varphi_a(\xi, s_n) - \varphi_a''(\xi, s_n) = 2h_3 \frac{s - s_n}{c} \delta(\xi - a) \varphi_a(\xi, s_n) + \frac{s^2 - s_n^2}{c^2} \varphi_a(\xi, s_n). \quad (\text{A.4})$$

Substituting Eq.(A.4) into Eq.(A.1) yields

$$\begin{aligned}
I &= \int_0^L \left( \frac{\varphi_a(x, s_n) \varphi_a(\xi, s_n)}{A_n(s - s_n)} + \text{Reg}(x, \xi, s_n) \right) \left( 2h_3 \frac{s - s_n}{c} \delta(\xi - a) \varphi_a(\xi, s_n) + \frac{s^2 - s_n^2}{c^2} \varphi_a(\xi, s_n) \right) d\xi \\
&\xrightarrow{s \rightarrow s_n} \varphi_a(x, s_n) \left[ \frac{2h_3}{A_n c} \int_0^L \delta(\xi - a) \varphi_a(L, s_n)^2 d\xi + \frac{2s_n}{A_n c^2} \int_0^L \varphi_a(\xi, s_n)^2 d\xi \right]
\end{aligned} \tag{A.5}$$

Combining Eqs.(A.3) and (A.5) finally leads to

$$A_n = \frac{2s_n}{c^2} \int_0^L \varphi_a(\xi, s_n)^2 d\xi + \frac{h_1}{c} \varphi_a(0, s_n)^2 + \frac{h_2}{c} \varphi_a(L, s_n)^2 + \frac{2h_3}{c} \varphi_a(a, s_n)^2. \tag{A.6}$$

This formula is derived in [1] with missing boundary terms because the authors overlook that the second term in Eq.(A.2) does not vanish as  $s \rightarrow s_n$ . After some algebra and simplifications the formula is given by

$$\begin{aligned}
A_n = -\frac{s_n}{c^2} \left\{ 4(L - a) \varphi(a, s_n)^2 h_3 + \left[ \left( 2 \left( L - \frac{1}{2} a \right) (h_1 - 1) e^{-\frac{s_n a}{c}} + 2(h_1 + 1) \left( L - \frac{1}{2} a \right) e^{\frac{s_n a}{c}} \right) \varphi(a, s_n) \right. \right. \\
\left. \left. + \frac{1}{2} a (h_1 - 1)^2 e^{-2 \frac{s_n a}{c}} - \frac{1}{2} a (h_1 + 1)^2 e^{2 \frac{s_n a}{c}} \right] h_3 + L(h_1^2 - 1) \right\}.
\end{aligned} \tag{A.7}$$

## Appendix B: Inhomogeneous boundary conditions

Consider the boundary value problem

$$\left( 2h_3 \frac{s}{c} \delta(x - a) + \frac{s^2}{c^2} \right) U(x, s) - \frac{d^2 U(x, s)}{dx^2} = w(x, s) \tag{B.1}$$

$$\frac{dU(0, s)}{dx} - \frac{h_1}{c} s U(0, s) = \gamma_1 \tag{B.2}$$

$$\frac{dU(L, s)}{dx} + \frac{h_2}{c} s U(L, s) = \gamma_2 \tag{B.3}$$

Let  $\tilde{U}(x, s)$  be the solution with  $\gamma_1 = \gamma_2 = 0$ . Then  $\tilde{U}(x, s) = \int_0^L G(x, \xi, s) w(\xi, s) d\xi$ , where  $G(x, \xi, s)$  is the Green's function of the homogeneous system. We wish to find a solution for general  $\gamma_1, \gamma_2$ .

The Green's function satisfies  $L[G(x, \xi, s)] = \delta(x - \xi)$  with  $L = \left(2h_3 \frac{s}{c} \delta(x - a) + \frac{s^2}{c^2}\right) - \frac{d^2}{dx^2}$  and the homogeneous boundary conditions. Consider the Green's function of the adjoint problem. Let  $L^* = \left(2h_3 \frac{\bar{s}}{c} \delta(x - a) + \frac{\bar{s}^2}{c^2}\right) - \frac{d^2}{dx^2}$  be the adjoint operator to  $L$  [12, chap 20.3]. The adjoint Green function  $g$  satisfies  $L^*[g(x, \xi, s)] = \delta(x - \xi)$  and the adjoint boundary conditions.

We begin with the integration by parts to obtain

$$\int_0^L \bar{g}(x, \xi, s) L[U(x, s)] dx = [\bar{g}(x, \xi, s) U_x(x, s) - \bar{g}_x(x, \xi, s) U(x, s)]_0^L + \int_0^L \overline{L^* g(x, \xi, s)} U(x, s) dx \quad (\text{B.4})$$

In view of the boundary conditions the first term on the right of Eq.(B.4), the so called surface term, simplifies to  $\bar{g}(L, \xi, s) \gamma_2 - \bar{g}(0, \xi, s) \gamma_1$ , while the second term becomes  $U(\xi, s)$  because of  $\delta(x - \xi)$ . This yields

$$U(\xi, s) = \int_0^L \bar{g}(x, \xi, s) L[U(x, s)] dx - \bar{g}(L, \xi, s) \gamma_2 + \bar{g}(0, \xi, s) \gamma_1 \quad (\text{B.5})$$

It is known [12, chap 20.3] that  $\bar{g}(x, \xi, s) = G(\xi, x, s)$ . Substituting this into Eq.(B.5) and interchanging  $x$  and  $\xi$  we finally obtain

$$U(x, s) = \tilde{U}(x, s) + \gamma_1 G(x, 0, s) - \gamma_2 G(x, L, s). \quad (\text{B.6})$$

## Appendix C: Vibratory response for $h_1 = h_2 = 1$

When  $h_1 = h_2 = 1$  it is possible to obtain a closed form solution for the vibratory response by substituting Eq.(42) into Eq.(47). Here we give the final expressions for the integrals from the second line of Eq.(47). To shorten the

formulas we set  $f(\xi) := u(\xi, 0)$  and  $g(\xi) := \dot{u}(\xi, 0)$ :

$$\begin{aligned} \frac{1}{c^2} \int_0^L \Gamma_t(x, \xi, t) f(\xi) d\xi &= \left[ -\frac{h_3}{2(1+h_3)} f(2a-x-tc) H\left(t - \frac{a-x}{c}\right) + \frac{1}{2} f(x-tc) \right. \\ &\quad \left. + \frac{1}{2} f(x+tc) H\left(\frac{a-x}{c} - t\right) + \frac{1}{2(1+h_3)} f(x+tc) H\left(t - \frac{a-x}{c}\right) \right] H(a-x) \\ &\quad + \left[ -\frac{h_3}{2(1+h_3)} f(2a-x+tc) H\left(t - \frac{x-a}{c}\right) + \frac{1}{2} f(x+tc) + \frac{1}{2} f(x-tc) H\left(\frac{x-a}{c} - t\right) \right. \\ &\quad \left. + \frac{1}{2(1+h_3)} f(x-tc) H\left(t - \frac{x-a}{c}\right) \right] H(x-a); \end{aligned}$$

$$\begin{aligned} \frac{1}{c^2} \int_0^L \Gamma(x, \xi, t) g(\xi) d\xi &= \left[ -\frac{h_3}{2(1+h_3)} \int_0^t g(2a-x-\tau c) H\left(\tau - \frac{a-x}{c}\right) d\tau + \frac{1}{2} \int_0^t g(x-\tau c) d\tau \right. \\ &\quad \left. + \frac{1}{2} \int_0^t g(x+\tau c) H\left(\frac{a-x}{c} - \tau\right) d\tau + \frac{1}{2(1+h_3)} \int_0^t g(x+\tau c) H\left(\tau - \frac{a-x}{c}\right) d\tau \right] H(a-x) \\ &\quad + \left[ -\frac{h_3}{2(1+h_3)} \int_0^t g(2a-x+\tau c) H\left(\tau - \frac{x-a}{c}\right) d\tau + \frac{1}{2} \int_0^t g(x+\tau c) d\tau \right. \\ &\quad \left. + \frac{1}{2} \int_0^t g(x-\tau c) H\left(\frac{x-a}{c} - \tau\right) d\tau + \frac{1}{2(1+h_3)} \int_0^t g(x-\tau c) H\left(\tau - \frac{x-a}{c}\right) d\tau \right] H(x-a); \end{aligned}$$

$$\begin{aligned} \frac{1}{c^2} \int_0^t \int_0^L \Gamma(x, \xi, t-\tau) p(\xi, \tau) d\xi d\tau &= \frac{1}{2c(1+h_3)} \left[ -h_3 \int_0^{t-\frac{2a-x-\xi}{c}} \int_0^a p(\xi, \tau) d\xi d\tau \right. \\ &\quad \left. + (1+h_3) \int_0^{t-\frac{x-\xi}{c}} \int_0^x p(\xi, \tau) d\xi d\tau + (1+h_3) \int_0^{t-\frac{\xi-x}{c}} \int_x^a p(\xi, \tau) d\xi d\tau + \int_0^{t-\frac{\xi-x}{c}} \int_a^L p(\xi, \tau) d\xi d\tau \right] H(a-x) \\ &\quad + \frac{1}{2c(1+h_3)} \left[ \int_0^{t-\frac{x-\xi}{c}} \int_0^a p(\xi, \tau) d\xi d\tau - h_3 \int_0^{t-\frac{x+\xi-2a}{c}} \int_x^a p(\xi, \tau) d\xi d\tau \right. \\ &\quad \left. + (1+h_3) \int_0^{t-\frac{x-\xi}{c}} \int_a^x p(\xi, \tau) d\xi d\tau + \int_0^{t-\frac{\xi-x}{c}} \int_a^L p(\xi, \tau) d\xi d\tau \right] H(x-a). \end{aligned}$$



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